FAILURE PROCESS MODELING FOR SYSTEMS WITH GENERAL REPAIRS

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Many methods on modeling failure processes of repairable systems have been developed. Based on different assumptions on the repairs, the models can be classified into three categories: models for minimal repair, models for general repair and models for perfect repair. In this paper, we will discuss a log linear model based on the general repair assumption. This model is integrated with the Kijima virtual age model to consider the repair effect on failure intensity.

Keywords: Log-linear model, Kijima Model, Virtual age

1. Introduction

The failure process of a repairable system can be described by the failure intensity function. Functions with different mathematic forms have been used in the past, such as power law, log-linear, s-shaped function, etc. [1, 2, 6, 10]. Methods that can model the effect of repairs have been integrated into some of the above mentioned functions. A great review is provided by [7]. In general, for a complex system, depending on the nature of the failures, the effect of a repair can be classified into three categories:

- Minimal repair: This is also called "as bad as old". Repair following a failure will restore the system to its status right before the failure.
- Perfect repair: This is also called "as good as new". Repair can be treated as replacement. The system will be treated as brand new after a repair.
- General repair: This is also called "better than old but worse than new". After repair, the system is restored to a status somewhere between the above two situations.

Modeling and parameter estimation methods for all the above three cases have been well developed and discussed. In recent years, a fair amount of attention has been paid to the modeling of systems with "general repair". In general, the effect of a "general repair" can be modeled in two ways in the failure intensity function:

- Reduce the failure intensity
- Reduce the age of the system

Lawless and Thiagarajah [9] has used a log-linear model with reduced failure intensity to model systems with general repair. Guo [6] used the operation time and the number of repairs to model the failure intensity and the effect of repairs. Age-reduced model is first introduced by Kijima [8] and used by Gasmi et. al [5], Yanez, et. al [12] and many others. In age-reduced models, the concept of "virtual age" is introduced and integrated in the models. For any given operation time *t*, there is a corresponding virtual age of the system associated with it. The failure intensity is determined by the virtual age instead of the operation time. Virtual age is a function of the operation time and the number of repairs. According to Kijima, there are two types of virtual age:

Kijima Model I:

Kijima model I assumes that the *n*th repair can remove the damage incurred between the (*n*-1)th and the *n*th failures; therefore it partially reduces the additional age of the system from x_n to qx_n. Accordingly, the virtual age after the *n*th repair becomes:

$$v_n = v_{n-1} + qx_n \qquad (1)$$

or,

$$v_n = q(x_1 + x_2 + \cdots + x_n) = qt_n \qquad (2)$$

where x_i is the time interval from the *i*-1th failure to the *i*th failure; t_n is the operation time when the *n*th failure occurs. q is a model parameter that represents the effect of the *n*th repair. In Eq.(1) and (2), it can be seen that q is assumed to be a constant value although it can be different for each repair.

<u>*Kijima Model II:*</u>

This model assumes that the *n*th repair will remove the cumulative damage from both current and previous failures. The *n*th repair modifies the virtual age that has been accumulated till to the repair time, i.e., $v_{n-1} + x_n$. Therefore,

$$v_n = q(v_{n-1} + x_n) \tag{3}$$

which is the same as

$$v_n = q(q^{n-1}x_1 + q^{n-2}x_2 + \cdots + x_n) \tag{4}$$

The basic idea of the virtual age models is to substitute the real time with the virtual age for failure intensity calculation.

Applying the virtual age to power law failure intensity functions has been discussed by Gasmi, et. al [5] and Yanez et. al [12]. In this paper, we will illustrate how to applying virtual age to the log-linear model. Modeling and parameter estimation will be discussed in detail in the following sections. Case study will also be provided. Similar to most of the published models, the repair time is ignored.

2. Mythology

2.1. *Modeling and Parameter Estimation*

The log-linear model is related to the famous proportional hazard model proposed by Cox [1]. The failure intensity is assumed to have the following form:

$$\lambda(t) = e^{\theta' z(t)} \tag{5}$$

where $z(t) = \left(z_1(t), \ldots, z_p(t)\right)'$ is a vector of functions that depends on time *t*. Usually it represents the operation condition of a system at time *t*. $\theta = \left(\theta_1, \ldots, \theta_p\right)'$ is the unknown model parameters. If the operation condition is constant and failure rate can be simply modeled by time *t*, Eq.(5) becomes:

$$\lambda(t) = e^{a+bt} \tag{6}$$

In this paper, we will use Eq. (6) as our base model for illustration.

In Eq. (6), time *t* is the cumulative operation time of the system. If the repair is minimal repair, *t* is also the system age. Therefore, we can use Eq.(6) directly to get the failure intensity at time *t*. However, if the repairs are assumed to be general repairs, we need to use $v(t)$, the virtual age or the age at time *t* in Eq.(6). $v(t)$ is calculated using the following equation:

$$v(t) = v_n + (t - t_n) \quad (7)$$

where v_n is the age right after the latest repair before time *t*; t_n is the failure time of the latest failure before time *t*. Using $v(t)$ to replace *t* in Eq. (6), we have:

$$\lambda(t) = e^{a+b\times v(t)} \quad (8)$$

We can see there are three parameters in the log-linear model with virtual age. They are *a*, *b* and *q*. We will use Maximum Likelihood Estimation (MLE) to estimate them. Let the duration from the (*i*-1)th failure to the *i*th failure to be x_i. For simple, let's first assume the repair is minimal repair. So the system age $v(t)$ is the same as the operation time *t*. The failure distribution for x_i is:

$$P\left(X_i \le x_i \mid T_i > t_{i-1}\right) = P\left(T_n \le t_i \mid T_i > t_{i-1}\right) = \frac{F(t_i) - F(t_{i-1})}{1 - F\left(t_{i-1}\right)} \quad (9)$$

$F(t_i)$ is the CDF for the cumulative failure time at the *i*th failure. It can be obtained using the following equation:

$$F(t) = 1 - R(t) = 1 - e^{-\int_0^t \lambda(t)dt} = 1 - e^{-\int_0^t e^{a+bt}dt} = 1 - e^{-\frac{1}{b}e^a\left(e^{bt}-1\right)} \quad (10)$$

For the case of general repair, the failure time t_i in Eq. (9) is replaced by the corresponding system age v_i. Therefore, the failure distribution for x_i becomes:

$$P\left(X_i \le x_i \mid T_i > t_{i-1}\right) = P\left(T_n \le t_i \mid T_i > t_{i-1}\right) = \frac{F(x_i + v_{i-1}) - F(v_{i-1})}{1 - F\left(v_{i-1}\right)} \quad (11)$$

Take the derivative respect to x_i from Eq. (11), we get the conditional pdf (probability density function) for x_i as:

$$f(x_i \mid v_{i-1}) = e^{a+bv_{i-1}+bx_i-\frac{1}{b}\left(e^{a+bv_{i-1}+bx_i} - e^{a+bv_{i-1}}\right)} \quad (12)$$

where v_{i-1} can be calculated using Eq.(2) or Eq. (4). Once the pdf is obtained for each observed failure time, the likelihood function can be calculated using:

$$L(data \mid a,b,q) = f(x_1) f(x_2 \mid v_1) \cdots f(x_n \mid v_{n-1}) R\left(T \mid v_n\right) \qquad (13)$$

where *T* is the end time of the test or operation and:

$$R\left(T \mid v_n\right) = \frac{R(v_n + T - t_n)}{R(v_n)} = e^{-\frac{1}{b}\left(e^{a+bv_n+bT-bt_n} - e^{a+bv_n}\right)}$$

Take the logarithm transform, Eq.(13) becomes:

$$\begin{aligned} Ln(L) = &\sum_{i=1}^{n} \left[a + bv_{i-1} + bx_i - \frac{1}{b}\left(e^{a+bv_{i-1}+bx_i} - e^{a+bv_{i-1}}\right)\right] \\ &- \frac{1}{b}\left(e^{a+bv_n+bT-bt_n} - e^{a+bv_n}\right) \end{aligned} \qquad (14)$$

Eq. (14) is for a single system. For multiple system with the same failure behavior, the log likelihood function is:

$$\begin{aligned} Ln(L) = &\sum_{l=1}^{k} \sum_{i=1}^{n_l} \left[a + bv_{l,i-1} + bx_{l,i} - \frac{1}{b}\left(e^{a+bv_{l,i-1}+bx_{l,i}} - e^{a+bv_{l,i-1}}\right)\right] \\ &- \frac{1}{b}\sum_{l=1}^{k}\left(e^{a+bv_{l,n_l}+bT_l-bt_{l,n_l}} - e^{a+bv_{l,n_l}}\right) \end{aligned} \qquad (15)$$

Eq. (5) can be used to solve parameter *a*, *b* and *q* by maximizing the log likelihood function.

2.2. ***Model Applications***

Once the model parameters are obtained, we can use them to predict the number of failures and the failure intensity at any give time *T*. In the case of minimal repair, the system age is the same as the system operation time. A closed form solution exists for number of failures at time *T*. It is:

$$N(T) = \int_0^T \lambda(t)dt = \int_0^T e^{a+bt}dt = \frac{1}{b}e^a(e^{bT} - 1) \qquad (16)$$

However, for the case of general repair, no closed form equation for the number of failures exists. This is because $\lambda(t)$ is not a continuous function any more.

As shown in Eq. (8), $\lambda(t)$ is determined by the virtual age $v(t)$ at time *t*. From Eq.(2), (4) and (7), it is also clear that in order to get $v(t)$, we need to know the exact failure time for each failure before time *t*. However, this is impossible for a given future time *t* since future failures haven't occurred yet. If we assume the system failure behavior is represented by the failure and repairs we have seen, we can use simulation to make prediction. The model used in the simulation is the model we have found from the failure data. From the simulation, the expected number of failures and the expected virtual time at time *t* can be found. The flow chart of the simulation is given in Figure 1.

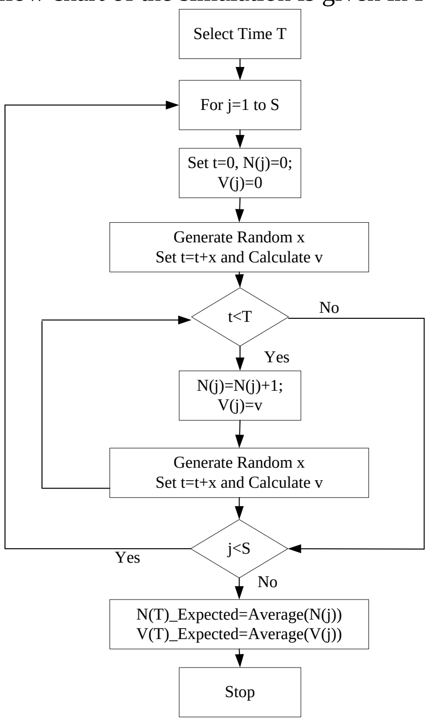

Figure 1. Simulation Flow Chart to Get the Expected N(T) and v(T)

In Figure 1, S is the number of simulations. Prediction using simulation is working through the following way. From each simulation, the simulated number of failures and the virtual age at time *T* are obtained. The expected $N(T)$ and $v(T)$ are the averages of the simulated $N(T)$ and $v(T)$ from S simulation runs. Using the expected $v(T)$, the expected value of the failure intensity at time *T* then can be calculated.

3. Case Study

The times to failure of a system is given in Table 1.

Table 1. Times to Failure Data

3444	12818	18131
5036	13212	19651
5260	15649	19703
5791	15683	22756
7604	16050	23093
8055	17045	24603
8493	17701	24953
9756	17882	25290

The log-linear function is used for the failure intensity. Two models are fitted to the data. One is the regular log-linear model with minimal repair assumption; the other is the log-linear model with Kijima type II virtual age for general repairs. The results are given in Table 2.

Table 2. Parameters for Log-linear Models

Parameter	Log-linear (minimal repair)	Log-linear (general repair)
a	-7.46507	-8.87644
b	0.000034	0.000418
q	-	0.813601
Log-likelihood	-190.931	-188.504

The log-likelihood values in Table 2 can be used to test if the parameter q is significant or not. If q is not significant, the following statistic will approximately follow a chi-square distribution:

$$LR = -2(\ln L_0 - \ln L_1) \sim \chi_1^2 \qquad (17)$$

where $\ln L_0$ and $\ln L_1$ are the log-likelihood value for the minimal repair model and the general repair model. Using the values are Table 2, we can get LR=4.855. The corresponding P-value for the given LR is 0.028. Therefore, Parameter q is significant at significance level of 0.05 since p-value is less than 0.05.

Applying the model parameters in the simulation (number of simulation = 5,000), we can simulate the expected number of failures, expected failure intensity and expected virtual age for any given time *T*. They are given in Figure 2, 3 and 4.

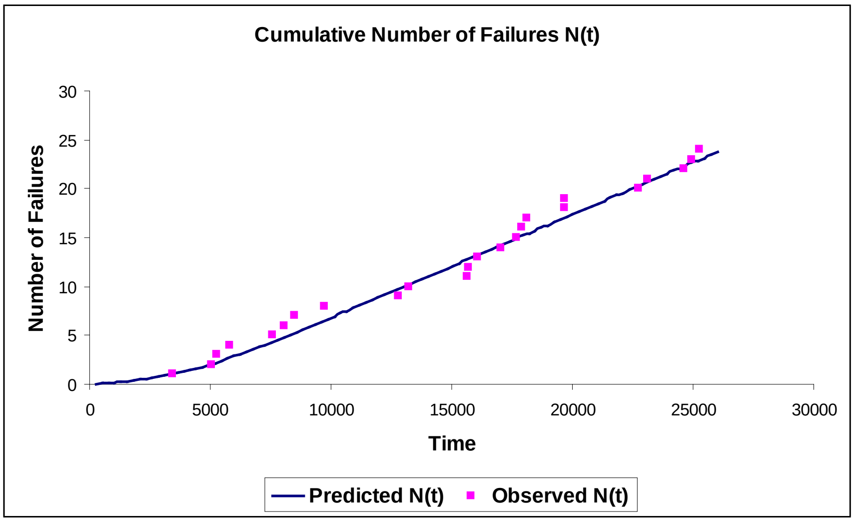

Figure 2. Failure Number N(T)

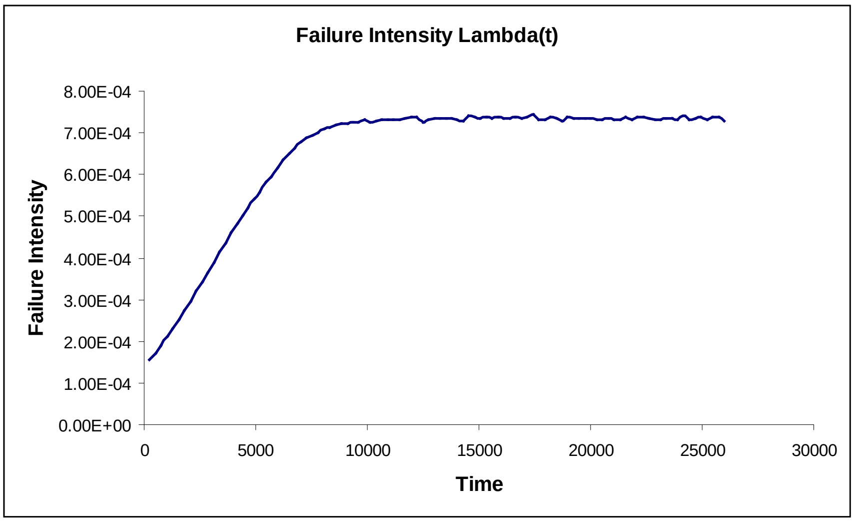

Figure 3. Failure Intensity Function $\lambda(T)$

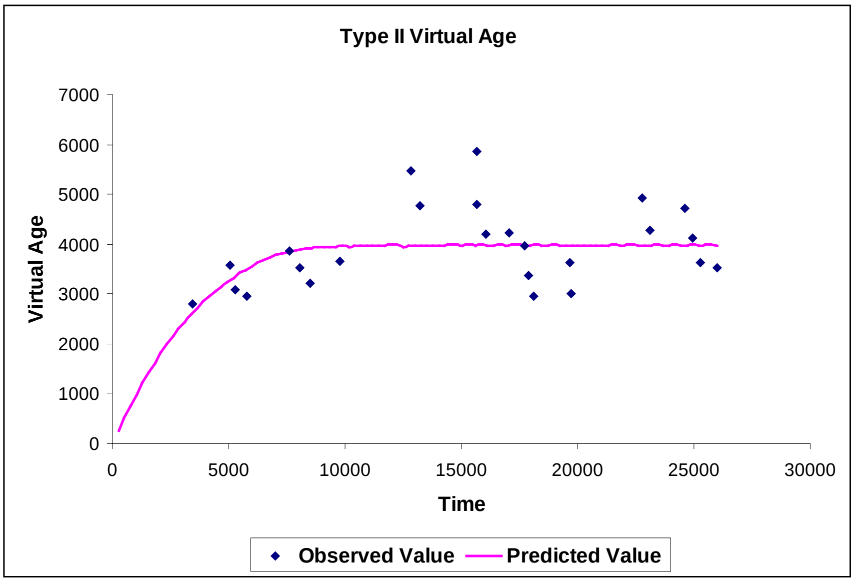

Figure 4. Kijima Type II Virtual Age $v(T)$

4. Conclusion

In this paper, we discussed how to integrate the virtual age idea into the existing log-linear function to model repairable systems with general repair. Parameter estimation and model applications are presented in detail. This model is useful when only information on the number of failures and repairs are available. If more detailed information such as the inspection strategy, severity of the failures, the effectiveness of each repair, and the operation condition of the system when failure occurs are available, more advanced models should be applied [3, 4, 11].

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