# A General Formula for Expected Number of Failures

Miklós Szidarovszky, HBM Prenscia

Ferenc Szidarovszky, PhD, Ridgetop Group Inc.

Sharon L. Honecker, PhD, HBM Prenscia

Key Words: Expected number of failures, analytic methods, integral equation

*SUMMARY & CONCLUSIONS*

An equipment is considered which is subject to repairable failures on the time interval $[0, t]$. At the time of each failure, repair is performed which requires length of time $T$ to perform, and as the result of each repair, the virtual age of the equipment decreases by time period $u$. Minimal repair is obtained by $u = 0$ and complete repair / replacement by setting $u = t$ (all age accrued up to the point of failure).

A mathematical model was derived for the expected number of failures with partial repairs which require a certain time period to perform. The model derived is a two-dimensional integral equation. It was also shown that in the special cases of minimal repairs and failure replacements the general model could be reduced to the well-known special formulas. A computer method was also suggested for finding solutions and a simple approximating rule was introduced to approximate the solutions. A Weibull numerical example illustrated the theoretical developments.

## 1 INTRODUCTION

Random failures are usually characterized by probabilistic methods. If a failure is non-repairable, then failure replacement is performed, which is usually more expensive than preventive replacement. If a failure is repairable, then repair is done in order to bring back the equipment to working condition. In the case of minimal repair the equipment is brought back to its condition before the failure occurred, so there is no change in the probabilistic characteristics such as cumulative distribution function (CDF), probability density function (PDF) of time of failures and the failure rate. In the case of replacement the virtual age of the equipment drops to zero, and in the case of partial repairs the virtual age can be decreased by a fixed constant or by a factor less than unity. In some cases however, the failure rate is decreased by a given factor implying that the expected number of failures in any future interval are decreased by the same factor. Except in the case of minimal repair the probabilistic feature of time to failure is altered accordingly. It is well known that in the case of minimal repairs the expected number of failures in interval $[0, t]$ is given as $M(t) = \int_0^t \rho(\tau)d\tau$, where $\rho(\tau)$ is the failure rate. In the case of replacements $M(t)$ satisfies the integral equation $M(t) = F(t) + \int_0^t M(t - S)f(S)dS$, which can be solved by using a numerical integration formula [7] or Laplace transform. There are formulas and other known solutions in the literature that do not consider the time length needed to perform the repairs when no degradation occurs, and the modified probabilistic characteristics in case of different types of partial repairs.

There is large amount of works dealing with the application of probability theory in reliability engineering as well as finding optimal maintenance, repair and replacement strategies [1-6]. The objective of this paper is not the determination of optimal strategies, its aim is to find the expected number of failures in any time interval during life cycle of the equipment, which is an important input data for the optimal scheduling and finding appropriate logistics for maintenance, repair and preventive replacement actions.

This paper develops as follows. A general mathematical model is first introduced and its reduction to other classical models will be shown. Then a computation method is suggested to solve the resulted integral equation. A simple procedure is also introduced to provide approximating solutions. A small numerical example illustrates the methodology with the assumption of Weibull failure time distribution. The last section concludes the paper.

## 2 MATHEMATICAL MODEL

Let $F(t)$, $f(t)$, $\rho(t)$ and $R(t)$ denote the CDF, PDF, failure rate and reliability function of time to first failure in an equipment. Partial repairs are assumed in cases of the failures. Each repair requires $T$ time periods to complete, and the virtual age of the equipment decreases by a constant $u > 0$. If the virtual age is below $u$, then after repair, the equipment will become as new. Assume that at a certain time the virtual age of the equipment is $v$ and it is operating. We will derive a general formula $E(v, \Delta t)$ for the expected number of failures in interval $(v, v+\Delta t)$. This expectation is independent of the calendar age of the equipment, since its future state of health is determined only by its virtual age. For the sake of simplicity assume that the calendar age of the equipment at the beginning of the interval is set to $v$. Let $v+S$ be the time of the first failure after time $v$, which has the conditional CDF:

$$F_v(S) = \frac{F(v+S)-F(v)}{1-F(v)} = \frac{F(v+S)-F(v)}{R(v)} \quad (1)$$

and the corresponding PDF is clearly $f_v(S) = \frac{f(v+S)}{R(v)}$. Assume first that $S$ is given. Let now $N(v, \Delta t)$ be the number of failures in interval $(v, v+\Delta t)$ with $S$ given. Then

$$E(X(v,\Delta t)|S) = \begin{cases} 0, & \text{if } \Delta t < S \\ 1, & \text{if } S < \Delta t < S+T \\ 1+E\big(0, \Delta t-(S+T)\big), & \text{if } \Delta t > S+T \text{ and } v+S<u \\ 1+E\big(v+S-u, \Delta t-(S+T)\big), & \text{if } \Delta t > S+T \text{ and } v+S>u \end{cases} \quad (2)$$

In the first case no failure can occur in interval $(v, v+\Delta t)$, since the first failure is at $v+S$. In the second case the first failure occurs in interval $(v, v+\Delta t)$ but no more failure is possible, since the failure repair cannot be completed before time $v+\Delta t$. If $\Delta t > S+T$, then after repair the virtual age of the equipment becomes zero, otherwise it becomes $v+S-u$. The repair is completed at time $v+S+T$, when the equipment is in operating condition. The length of interval $(v+S+T, v+\Delta t)$ is $\Delta t-(S+T)$. In the first case $S>\Delta t$, in the second case $\Delta t - T < S < \Delta t$, in the third case $S < \min\{u-v, \Delta t - T\}$ and in the last case $u-v < S < \Delta t - T$. Here, we assumed that $\Delta t > T$. In the case of $\Delta t \le T$ only one failure can occur with probability $F_v(\Delta t)$, so $E(v,\Delta t) = F_v(\Delta t)$. By using the method of expectation by conditioning, we can have the following relation for $\Delta t > T$. If $u - v < \Delta t - T$, then

$$E(v,\Delta t) = \int_{\Delta t - T}^{\Delta t} f_v(S)dS + \int_{u-v}^{\Delta t - T}\big(1+E(v+S-u, \Delta t - S - T)\big)f_v(S)dS + \int_0^{u-v}\big(1+E(0,\Delta t - S - T)\big)f_v(S)dS \quad (3)$$

Otherwise

$$E(v,\Delta t) = \int_{\Delta t - T}^{\Delta t} f_v(S)dS + \int_0^{\Delta t - T}\big(1+E(0,\Delta t - S - T)\big)f_v(S)dS \quad (4)$$

In summary, $E(v, \Delta t)$ can be expressed as equation (5).

We next consider the two known special cases. First assume minimal repair with $u = T = 0$. Let $v = 0$, then only the second case occurs with $F_v(\Delta t) = F(\Delta t)$ and $\Delta t = t$. Then we have

$$M(t) = E(0,t) = F(t) + \int_0^t E(S, t-S) f(S)dS,$$

and since $E(S, t-S) = M(t) - M(S)$ we have

$$M(t) = F(t) + \int_0^t \big(M(t) - M(S)\big) f(S)dS$$

$$= F(t) + M(t)F(t) - \int_0^t M(S)f(S)dS$$

which is well known.

Assume next that failure replacements are done with $\Delta t = t$, $u = v + S$, and $v = T = 0$. Then, only the second case of (5) can occur, so

$$M(t) = E(0,t) = F(t) + \int_0^t E(0, t-S)f(S)dS = F(t) + \int_0^t M(t-S)f(S)dS$$

which is again the well-known integral equation.

## 3 AN APPROXIMATION

In the second and third cases of (5) in the integrand we have the term $E(v+S-u, \Delta t - S - T)$, which refers to an interval with starting point $v+S-u$ and length $\Delta t - S - T$. So, the endpoint of the interval is $v+S-u+\Delta t - S - T = v + \Delta t - u - T$. This expectation is based on the condition that at the starting point of the interval the equipment is in working condition. If we ignore this condition, then the expectation equals $M(v+\Delta t - u - T) - M(v+S-u)$. By selecting $v = 0$, only the second case of (5) is impossible, so (5) can be written as equation (6).

The second case of (6) can be rewritten as

$$F(t) + \int_0^u M(t-S-T)f(S)dS + M(t-u-T)[F(t-T) - F(u)] - \int_u^{t-T} M(S-u)f(S)dS$$

Model (6) can be considered as an approximation of model (5).

$$E(v,\Delta t)=\begin{cases} F_v(\Delta t), & \text{if} \quad \Delta t \le T, \text{otherwise} \\ F_v(\Delta t)+\int_0^{\Delta t-T} E(v+S-u,\Delta t-S-T) f_v(S)dS\,, & \text{if} \quad u-v\le 0 \\ F_v(\Delta t)+\int_0^{u-v} E(0,\Delta t-S-T) f_v(S)dS+\int_{u-v}^{\Delta t-T} E(v+S-u,\Delta t-S-T) f_v(S)dS\,, & \text{if} \quad 0<u-v<\Delta t-T \\ F_v(\Delta t)+\int_0^{\Delta t-T} E(0,\Delta t-S-T) f_v(S)dS\,, & \text{if} \quad u-v\ge \Delta t-T \end{cases} \tag{5}$$

$$M(t)=\begin{cases} F(t), & \text{if} \quad t\le T, \text{otherwise} \\ F(t)+\int_0^{u} M(t-S-T) f(S)dS+\int_u^{t-T}\big(M(t-u-T)-M(S-u)\big) f(S)dS\,, & \text{if} \quad 0<u<t-T \\ F(t)+\int_0^{t-T} M(t-S-T) f(S)dS\,, & \text{if} \quad u\ge t-T \end{cases} \tag{6}$$

## 4 COMPUTER METHODS

Consider the $(v,\Delta t)$ space for $v,\Delta t\ge 0$, and place a uniform grid on it with a small step size $h$ in both directions. If $T=Nh$, then for all $v\ge 0$, the value of $E(v,\Delta t)$ can be computed by equation (1) with $\Delta t \le T$. So the function value of $E(v,\Delta t)$ will be known for the first $N+1$ rows of the grid. We can sequentially compute the function values for row $N+2$, and then for row $N+3$, and so on until $\Delta t$ reaches the required length. Let now $\Delta t=Kh$ $(K>N+1)$. Notice that in the integrals of (5) the second argument of $E$ is $\Delta t-S-T$ which is less than $Kh$ by at least $Nh$, so the corresponding values of function $E$ are in earlier rows, which were already computed. Moving from row $N+2$ up sequentially we are able to find all values of $E$ at all grid-points of the actual row. The final result is $E(0,\Delta t)$ which is the same as $M(\Delta t)$. The expected number of failures in any interval $(t_1,t_2)$ is $M(t_2)-M(t_1)$, which is not necessarily equal to $E(t_1,t_2-t_1)$, since at time $t_1$ the equipment can be in working condition or under repair.

The approximating model can be solved numerically in a much easier way, since only a one-dimensional grid is needed in the $t\ge 0$ line and we can compute the values of $M(t)$ in the order of $t=0,h,2h,\ldots$.

## 5 NUMERICAL EXPERIMENT

An equipment was considered that follows a Weibull distribution. The parameters were selected as $\beta=1.1$ and $\eta=1.5$ time periods. It was assumed that each repair decreases the virtual age of the equipment with $u=0.03$ time periods and each repair needs $T=0.1$ periods to complete.

In our computation, we used the complete model (5), its approximation model (6), as well as simulations. The results are presented in Table 1.

Table 1: Computed values of $M(t)$

| t | $M(t)$ by method (5) | $M(t)$ by method (6) | $M(t)$ by simulation |
|---|---|---|---|
| 1 | 0.5980 | 0.5969 | 0.5924 |
| 2 | 1.2745 | 1.2667 | 1.2794 |
| 3 | 1.9807 | 1.9632 | 1.9917 |
| 4 | 2.7112 | 2.6726 | 2.7168 |
| 5 | 3.4596 | 3.3866 | 3.4725 |

In models (5) and (6) the trapezoidal formula was used to compute the integrals with step size $h=0.01$, as the error for the trapezoidal formula is on the order of $O(h^2)$. In the simulation, 100,000 repeats were used to minimize error. The results by method (5) and simulation differ by less than 1%, which is very good. Method (6) has larger errors, with the error growing the further in time one estimates, but still acceptable (within a few percent) for the case study values. Notice that solution of model (6) is much easier to compute than that of the "exact" model (5).

## ACKNOWLEDGEMENTS

The authors would like to thank Peyman Siyari for his help on numerical experiments.

## REFERENCES

[1] M. Hamidi, , F. Szidarovszky and M. Szidarovszky, "New One Cycle Criteria for Optimal Preventive Replacement Policies," *Reliability Engineering and System Safety*, 154, 42-48, 2016.

[2] A. Jardine and A. Tsang, "*Maintenance, Replacement, Reliability: Theory and Applications*", Boca Raton: CRC Press, 2013.

[3] T. Nakagawa, "*Advanced Reliability Models and Maintenance Policies,*" London: Springer, 2008.

[4] T. Nakagawa, "*Random Maintenance Policies,*" London: Springer, 2014.

[5] T. Nakagawa and X. Zhao, "*Maintenance Overtime Policies in Reliability Theory: Models with Random Working Cycles,*" London: Springer, 2015.

[6] S. Osaki, "*Stochastic Models in Reliability and Maintenance,*" London: Springer, 2002.

[7] F. Szidarovszky and S. Yakowitz, "*Principles and Procedures of Numerical Analysis,*" New York: Plenum, 1978.

*BIOGRAPHIES*

Miklós Szidarovszky
1450 S. Eastside Loop
Tucson, AZ 85710-6703, USA

e-mail: Miklos.Szidarovszky@hbmprenscia.com

Miklós Szidarovszky is a Technologist at HBM Prenscia. He is involved in the development of various ReliaSoft software products, the delivery of training seminars, and consulting projects. His areas of interest include Life Data Analysis, Accelerated Testing, System Reliability, Probabilistic Event and Risk Analysis, and Risk Based Inspection. Mr. Szidarovszky holds a B.S. and an M.S. in Chemical Engineering from the University of Arizona. He is a Certified Reliability Professional (CRP) and a Certified Scrum Master (CSM).

Ferenc Szidarovszky
3580 W. Ina Road
Tucson, AZ 85741, USA

e-mail: szidarka@gmail.com

Ferenc Szidarovszky is a senior researcher with the Ridgetop Group Inc. working on theoretical issues concerning reliability, robustness, optimal maintenance and replacement policies and prognostics in complex systems. In addition to these industrial applications he also performs research in systems theory, especially in dynamic economic systems. He has earned two PhD degrees, one in Applied Mathematics and another in Economics in Hungary. The Hungarian Academy of Sciences has awarded him with two higher degrees, Candidate in Mathematics and Doctor of Engineering Sciences. After graduating in Hungary, he became a professor of the Eotvos University of Sciences, Budapest and later became an acting department head at the Budapest University of Horticulture and Food Industry. In 1987 he moved to the US and from 1988 he was a professor of the Systems and Industrial Engineering Department at the University of Arizona. From September 2011 he was a Senior Researcher at ReliaSoft Corporation, and as of July 2015 he is with Ridgetop Group. As a Hungarian professor he was an investigator in the first US-Hungarian joint research project jointly supported by the NSF and Hungarian Academy of Sciences on Decision Models in Water Resources Management. He has performed research in very broad fields of applied mathematics including numerical analysis, optimization, multi-objective programming, game theory, dynamic systems, applied probability and their applications in industrial processes, environment, water resources management, economics, and computer network security among others. He is the author of 21 books and over 350 articles in international journals in addition to numerous conference presentations and short courses taught in many countries. He currently also has a professor position with the Applied Mathematics Department of the University of Pecs, Hungary, where his duty is a one-week long intensive Game Theory Course every year.

Sharon L Honecker
1450 S. Eastside Loop
Tucson, AZ, 85710-6703, USA

e-mail: sharon.honecker@hbmprenscia.com

Dr. Sharon Honecker is the Director of Technology for ReliaSoft Products at HBM Prenscia. For the last decade, Sharon has worked in the field of reliability engineering, providing training and consulting services to customers in industries including automotive, aerospace, oil and gas, and national laboratories. She served as the chair of an IEEE standards committee responsible for writing and maintaining reliability standards geared toward the nuclear power industry. In addition to her work in the field of reliability, Sharon has experience in low-cycle fatigue, fluid mechanics, and mechanical testing and she has authored papers in fatigue, microfluidics, and reliability. Sharon holds a PhD in mechanical engineering from the University of Illinois at Urbana-Champaign.