Reliability Allocation and Optimization for Complex Systems

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Key Words: Allocation, Reliability Allocation, Allocation Algorithm, Reliability Optimization, System Reliability, System Design

SUMMARY & CONCLUSIONS

During the design phase of a product, reliability engineers are called upon to evaluate the reliability of the system. The question of how to meet a reliability goal for the system arises when the estimated reliability is inadequate. This then becomes a reliability allocation problem at the component level. In this paper, a general model estimates the minimum reliability requirement for multiple components within a system that will yield the goal reliability value for the system. The model consists of two parts. The first part is a nonlinear programming formulation of the allocation problem. The second part is a cost function formulation to be used in the nonlinear programming algorithm. A general behavior of the cost as a function of a component’s reliability is assumed for this matter. The system’s cost is then minimized by solving for an optimum component reliability, which satisfies the system’s reliability goal requirement. Once the reliability requirement for each component is estimated, one can then decide whether to achieve this reliability by fault tolerance or fault avoidance. The model has yielded very encouraging results and it can be applied to any type of system, simple or complex, and for a variety of distributions. The advantage of this model is that it is very flexible, and requires very little processing time. These advantages make the proposed reliability allocation solution a great system design tool. A computer program has been developed and the model is available in a commercial software package called BlockSim™.

1. INTRODUCTION

The problem of reliability allocation and optimization has been widely treated by many authors. Although most of the attention to this issue has been given to the redundancy allocation problem (Refs. 5,6,8), a different approach to the problem is taken in this paper. Instead of concentrating exclusively on redundancy allocation, the minimum required reliability for each component of a system will be estimated in order to achieve a system reliability goal with minimum cost. Thereafter, the engineer can decide whether this minimum required component reliability will be achieved via fault avoidance or redundancy. The model allocates reliability to a component according to the cost of increasing its reliability. The most costly components (with cost representing volume, cost, weight or any other quantity of concern) will be assigned the lowest increases in reliability. Several methods for addressing this type of allocation problem are available. The majority of them however, are limited in their application to simple systems consisting of exponential units. In this paper the allocation problem is formulated as a constrained nonlinear optimization problem (Ref. 8), and solved using an algorithm based on the Quasi-Newton and Lagrange multipliers method. With this approach reliability can now be allocated to the components of any type of system, complex or not, and for a mixture of failure distributions for the components of the system. Although the nonlinear programming formulation to the problem has been proposed in the past by several authors (Refs. 1, 5, and 8), and one would expect this approach to be widely used among reliability engineers, very little attention has been given to its implementation. Two major factors have contributed to this situation. First, the model requires the system’s analytical reliability equation as an input. Although this poses no major problem in simple systems, it can become quite a challenge (and very time consuming) in complex systems. Second, the model also requires cost as a function of the component’s reliability as an input, and this is not always available to engineers. This problem has been addressed by authors (Refs. 1 and 8) through the introduction of general mathematical formulations for the required cost functions. These mathematical formulations depend on certain parameters that must be supplied by the engineers. Quantifying these parameters has not been an easy task in many instances, however, since a number of them are constants with no close relation to reliability principles. These shortcomings in attempts to formulate the allocation problem as a nonlinear programming problem have been resolved in this paper. A breakthrough in the implementation of the model was achieved through the introduction to the general public of a system reliability and maintainability analysis software package, BlockSim™. This software provides the system’s (simple or complex) analytical reliability equation. This equation can now be imported directly as an input to the optimization algorithm. Secondly, the cost function problem is addressed through the proposal of a general cost function, which is a function of parameters that can easily be quantified by engineers, and is simple in its use.

The allocation problem addressed in this paper is of great practical importance. Reliability engineers are often called upon to make decisions as to whether to improve a certain component or components in order to achieve a minimum required system reliability. For example, consider a system consisting of three components connected reliability-wise in series. The reliability of each of the components is 0.7, 0.8, and 0.9 respectively. Under the independence assumption, the reliability of the system will be 0.504. Assuming that a system reliability performance of 0.8 were sought, the current design is clearly inadequate. The question now becomes one of how this goal can be reached. The reliability goal cannot be reached by increasing the reliability of just one component.
One might then suggest increasing the reliability of two components. The question then becomes “which two components?” Should the reliability of all three components be improved? How feasible is it to improve a component’s reliability? Figure 1 is an illustration of this typical example of a decision-making dilemma. In order for these questions to be answered, another quantity is considered: cost. The challenge then becomes to model the cost as a function of reliability. The preferred approach would be to formulate the cost function from actual cost data. In many cases however, this data is not available and is hard to obtain. This problem is addressed with the introduction of a general mathematical formulation for the cost function, which is assumed to have an exponential behavior. This function will act as the penalty of increasing a component’s reliability. The overall system cost, which is the objective function to be minimized, is assumed to be the summation of each component’s cost.

Consider a system consisting of \( n \) components. A goal reliability is sought for this system. The objective is to allocate reliability to all or some of the components of that system, in order to meet that goal with a minimum cost. The problem is formulated as a nonlinear programming problem as follows:

\[
P: \quad \min \quad C = \sum_{i=1}^{n} c_i(R_i) \\
\text{s.t.} \quad R_i \geq R_G \\
R_{i,\text{min}} \leq R_i \leq R_{i,\text{max}}, \quad i = 1, 2, ..., n.
\]

This formulation is designed to achieve a minimum total system cost, subject to \( R_G \), a lower limit on the system reliability. The next step will be to obtain the system’s analytical reliability function (in terms of its component’s reliability). Several methods exist for obtaining the system’s reliability equation, a review of them can be found in Ref. 3. In this paper the recently available BlockSim software will be used, which is designed to solve for the system’s analytical reliability function.

The next step is to obtain a relationship for the cost of each component as a function of its reliability. An empirical relationship can be derived based on past experiences and/or data for similar components. In many cases however, such data is not available. In order to overcome this problem, a general behavior for the cost function is proposed in this paper, as follows:

\[
c_i(R_i; f_i, R_{i,\text{min}}, R_{i,\text{max}}) = e^{(1-f_i) \frac{R_i-R_{i,\text{min}}}{R_{i,\text{max}}-R_{i,\text{min}}}} \quad (2)
\]

This is an exponential behavior, as shown in Figure 2, and it contains three parameters, namely, \( f_i, R_{i,\text{min}}, \) and \( R_{i,\text{max}}. \) The first parameter, \( f_i, \) is the feasibility of increasing a component’s reliability, and it assumes values between 0 and 1. The second parameter, \( R_{i,\text{min}}, \) is the initial (current) reliability value of the \( i \)th component obtained from the failure distribution of that component and for the specified time. The third parameter, \( R_{i,\text{max}}, \) is the maximum achievable reliability of the \( i \)th component. For example, in Figure 2, the initial reliability for that particular component is 70%, the maximum achievable reliability is 99%, and the cost function is plotted for a 0.1 feasibility. The behavior of eq (2) with respect to each of the parameters, as well as the detailed explanation of the meaning of these parameters is presented in Sections 4.1 and 4.2.

The proposed cost function given by eq (2) satisfies the following requirements (Ref 1):

1. Cost is a monotonically increasing function of component reliability.
2. Cost of a high reliability component is very high.
3. Cost of a low reliability component is very low.

**2. NOTATION**

- **\( C \)**: total system cost,
- **\( c_i(R_i) \)**: cost of component/subsystem \( i, \)
- **\( R_i \)**: reliability of component/subsystem \( i, \)
- **\( n \)**: number of components within the system considered in the optimization,
- **\( R_{i,\text{min}} \)**: minimum reliability of component/subsystem \( i, \)
- **\( R_{i,\text{max}} \)**: maximum achievable reliability of component/subsystem \( i, \)
- **\( R_s \)**: system reliability
- **\( R_G \)**: system reliability goal
- **\( f_i \)**: feasibility of increasing the reliability of component/subsystem \( i, \)
- **\( I_R \)**: reliability importance
- **\( \beta \)**: Weibull shape parameter
- **\( \eta \)**: Weibull scale parameter

**3. ASSUMPTIONS**

1. The system’s reliability function can be obtained.
2. All systems consist of \( s \)-independent components/subsystems.
3. The system and its components/subsystems can only assume two states, failed and operational.
4. Derivative of cost (with respect to reliability) is a monotonically increasing function of reliability.

It can be seen that the cost function of eq. (2) is easy to implement, with only two required inputs (in addition to the failure distribution of the component), namely the feasibility and the maximum achievable reliability. Note that this penalty function (eq. (2)) is dimensionless. It essentially acts as a weighting factor that describes the difficulty in increasing the component reliability from its current value. The application of the model is illustrated for a series and complex system in Section 5.

Examining the cost function given by eq. (2) yields the following observations:
1. The cost increases as the allocated reliability departs from the minimum reliability (current value of reliability), and approaches the maximum achievable reliability.
2. The cost is a function of the range of improvement, which is the difference between the component’s initial reliability and the corresponding maximum achievable reliability.
3. The exponent in eq. (2) approaches infinity as the component’s reliability approaches its maximum achievable value. This means that it is easier to e.g., increase the reliability of a component from 70% to 75% than increasing its reliability from 95% to 96%.

4.1 Feasibility
The feasibility parameter is a constant, which represents the difficulty in increasing a component/subsystem’s reliability relative to the rest of the components in the system. Depending on the design complexity, technological limitations, etc., certain components can be very hard to improve, relative to other components in the system. Clearly, the more difficult it is to improve the reliability of the component/subsystem, the greater the cost. Examining the effect of the feasibility on the cost function of eq. (2), it can be seen that the lower the feasibility value, the more rapidly the cost function approaches infinity.

4.2 Maximum Achievable Reliability
The maximum achievable reliability is a limiting reliability value. For example, a reliability of 100% is a limiting reliability. However, technological or financial constraints might dictate a maximum achievable reliability for certain components/subsystems other than 100%. For this reason the maximum achievable reliability is incorporated in eq. (2) as one of the parameters. The maximum achievable reliability acts as a scale parameter for the cost function. By decreasing $R_{\text{max}}$, the cost function is compressed between $R_{\text{min}}$ and $R_{\text{max}}$, as shown in Figure 4.

4.3 Other forms of the cost function
The proposed cost function in this paper, given in eq (2), represents a general behavior of the cost as function of reliability, to be used in the case where an actual function is
not available. Authors (Ref. 3,4) have suggested other general forms as well. It is suggested, however, that these general functions are used individually within a system and do not get mixed with other cost functions. For example, it is not recommended to use a cost equation such as the one given by eq. (2) for some of the components in the system and a different cost equation for the rest of the components, particularly when these functions do not represent actual costs. Empirical forms for the cost function can also be derived based on past data, or models can be fitted on cost data obtained from the development phase of the product. Since the allocation problem has been formulated as nonlinear programming other forms for the cost function can be used. In BlockSim™, the engineer has the flexibility of using the cost function given by eq. (2), or any other user-defined cost function.

5. APPLICATIONS

The following examples illustrate the steps and procedure for solving the allocation problem. The model described in Section 4 has been coded, and a nonlinear programming algorithm is used for its solution. The algorithm has been implemented as part of BlockSim™. BlockSim™ is used to solve the following examples.

5.1 Application to a Series System

Consider a system consisting of three components connected reliability-wise in series. Assume the objective reliability for the system is 90% for a mission time of 100 hrs.

The first step is to obtain the system’s reliability equation. In this case, and assuming independence, the reliability of the system, $R_s$ is given by:

$$R_s = R_1 \cdot R_2 \cdot R_3$$

Eq. (1) can now be written as:

$$P: \min C = \sum_{i=1}^{3} c_i (R_i)$$

s.t. $R_1 \cdot R_2 \cdot R_3 \geq R_G$

$$R_{i,\min} \leq R_i \leq R_{i,\max}, \quad i = 1, 2, 3.$$

Using the cost function given by eq. (2) to represent the behavior of the cost as a function of reliability for each individual component, eq. (4) can be rewritten as follows

$$P: \min C = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} c_{ij} (R_i) \right)$$

s.t. $R_1 \cdot R_2 \cdot R_3 \geq R_G$

$$R_{i,\min} \leq R_i \leq R_{i,\max}, \quad i = 1, 2, 3.$$

Five scenarios will be considered for the allocation problem.

**Case 1:** All three components are identical whose time-to-failure are described with a Weibull distribution with $\beta = 1.318$ and $\eta = 312$ hrs. All three components have the same feasibility value.

**Case 2:** Same as in Case 1, but component 1 has a feasibility of 0.9, component 2 a feasibility of 0.5 and component 3 a feasibility of 0.1.

**Case 3:** Component 1 has 70% reliability, component 2 has 80% reliability, and component 3 has 90% reliability, all for a mission duration of 100 hrs. All three components have the same feasibility value of 0.9.

**Case 4:** Component 1 has 70% reliability and 0.9 feasibility, component 2 has 80% reliability and a 0.5 feasibility, and component 3 has 90% reliability and a 0.1 feasibility, all for a mission duration of 100 hrs.

**Case 5:** Component 1 has 70% reliability and 0.1 feasibility, component 2 has 80% reliability and a 0.9 feasibility, and component 3 has a 90% reliability and a 0.5 feasibility, all for a mission duration of 100 hrs.

In all cases the maximum achievable reliability, $R_{i,\max}$ for each component is 99.9% for a mission duration of 100 hrs.

First, Case 1 is considered. In this scenario, the components are identical with

$$R_i(100) = e^{-\left(\frac{100}{312}\right)^{1.318}} = 0.8.$$

This reliability value (the initial reliability) corresponds to the minimum reliability, $R_{i,\min}$. Using eq. (2), the optimization algorithm in BlockSim, and the specified parameters of this case, the resulting optimal reliability allocation for each component is

$$R_1(100) = R_2(100) = R_3(100) = 96.55\%.$$

In other words, each component’s reliability should be at least 96.55% at 100 hrs in order for the system’s reliability to be 90% at 100 hrs. This result was expected since the components are identical, thus all components will be assigned the same reliability.

The results for Cases 1 through 5 are summarized in Table 1.

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Component 1</td>
<td>0.9655</td>
<td>0.9874</td>
<td>0.9552</td>
<td>0.9790</td>
</tr>
<tr>
<td>Component 2</td>
<td>0.8653</td>
<td>0.9649</td>
<td>0.9553</td>
<td>0.9884</td>
</tr>
<tr>
<td>Component 3</td>
<td>0.9655</td>
<td>0.9463</td>
<td>0.9765</td>
<td>0.9824</td>
</tr>
</tbody>
</table>

Examining the results for Case 2, it can be seen that the highest reliability was allocated to component 1 with the higher feasibility value. The lowest reliability was assigned to component 3 with the lowest feasibility value.

In Case 3 the components were different, but had the same feasibility values. In other words, all three components have the same opportunity for improvement. This case differs from Cases 1 and 2 since there are two factors, not present previously, which will affect the outcome of the allocation in this case. First, each component in this case has a different reliability importance (impact of a component on the system’s reliability), where in Cases 1 and 2 all three components had the same reliability importance. The reliability importance of component $i$, is given by [Ref. 4]

$$I_R(i) = \frac{\partial R_s}{\partial R_i}.$$
The partial derivatives of eq. (3) with respect to each component’s reliability were calculated, and the results are plotted in Figure 5, where it can be seen that component 1 has the greatest reliability importance and component 3 the smallest (this reliability importance also applies in Cases 4 and 5). This indicates that the reliability of component 1 should be significantly increased, since it has the biggest impact on the overall system reliability.

In addition, each component’s cost in Case 3 also depends on the difference between each component’s minimum reliability and its corresponding maximum achievable reliability. (In Cases 1 and 2 this was not an issue, because the components were identical.) The greater this difference is, the greater the cost of improving the reliability of a particular component will be, relative to the other two components. This difference between the initial reliability of a component and its maximum achievable reliability is the range of improvement for that component. Since all three components have the same maximum achievable reliability, component 1 is the most cost efficient component to improve (it has the largest range for improvement), as it is illustrated in Figure 6. At the same time, however, there is a reliability value between the initial and the maximum achievable reliability, beyond which it becomes cost restrictive to improve any further. This reliability value is dictated by the feasibility value. From Table 1, it can be seen that in Case 3 there was a 25.52% improvement for component 1, a 16.49% for component 2, and a 7.65% for component 3. On the other hand in Case 4, component 1 was assigned an even greater increase of 27.9%, with components 2 and 3 receiving a lesser increase than in Case 3, of 15.53% and 6.24% respectively. This is due to the fact that component 1 has the largest feasibility value of 0.9, and component 3 the lowest of 0.1, which means that it is more difficult to increase component 3 than component 1.

Finally, in Case 5, where the feasibility values are reversed with component 1 having the smallest value of 0.1 and component 3 the largest of 0.9, the increase on component 1’s reliability is less compared to the increase for Cases 3 and 4. Note however, that components 2 and 3 still received a smaller increase in reliability than component 1, since their ranges of improvement are smaller. Clearly, component 3 was assigned the smallest increase in reliability in Cases 3, 4, and 5, because its initial reliability is very close to its maximum achievable reliability.

5.2 Application to a Complex System

Consider the system shown in Figure 7. All components have the same initial reliability of 90% at a given time, and the same feasibility value of 0.9. A system reliability goal of 90% (at the same given time for the components) is sought. The equation for the system reliability obtained from BlockSim™, is given by,

\[
R_S = R_1R_2R_3R_4R_6R_7 + R_1R_3R_6R_7 + R_3R_4R_6R_7 + R_1R_2R_4R_6R_7 - R_1R_3R_4R_6R_7 - R_3R_4R_6R_7 + R_1R_2R_3R_5R_7 + R_1R_2R_3R_6R_7 + R_1R_2R_3R_5R_6R_7 + R_1R_2R_3R_7 + R_1R_2R_3R_5R_7 - R_1R_2R_3R_6R_7 - R_1R_2R_3R_5R_6R_7 + R_1R_2R_3R_7
\]

Prior to solving the reliability allocation problem, a preliminary analysis can be performed in order to assess the outcome of the solution. This can be achieved by calculating the reliability importance of each component of the system. Using eq. (6), the reliability importance was calculated and the results are plotted in Figure 8.
impact of each component on the system reliability due to its position in that system is quantified with the reliability importance. Therefore, it is expected that components with high reliability importance will be assigned a high reliability, and components with the same importance will be assigned the same reliability value. In Figure 8 it can be seen that components 1 and 7 are the most critical with the same importance value. Subsequently, it is expected that the system reliability will be primarily allocated to components 1 and 7 with the same reliability value. Similarly, for the rest of the components. The results for two different maximum achievable reliability values are summarized in Table 2. In this table it can be seen that the results from the optimization are consistent with the preliminary assessment based on the reliability importance of the components. In addition, the effects of the maximum achievable reliability on the optimization solution can also be observed. In particular, for $R_{\text{max}}=0.999$, the reliability is primarily allocated to Components 1 and 7, with no improvement for Components 2, 3, and 4, and very a small improvement for Components 5 and 6. On the other hand for $R_{\text{max}}=0.96$, the reliability of Components 1 and 7 is improved up to the point where it becomes cost restrictive (0.9515). Thereafter, the allocation process is concentrated on the rest of the components in the system.

This example illustrates that the complexity of the system’s reliability equation can increase, but at the same time the allocation problem can still be solved successfully. In the same manner, systems with redundancy and standby components can also be optimized as long as the system’s reliability equation can be obtained analytically in the form of eq. (7).

6. CONCLUSIONS

In this paper the system reliability optimization problem through reliability allocation at the component level was examined. The problem was approached as a nonlinear programming problem and a general cost equation was suggested. This cost function is easy to use since it is simple in its form, with parameters that can be easily quantified. The nonlinear programming part of the model can also be used for other cost functions. Further research should be concentrated in obtaining such functions based on actual cost data. The advantage of the model is that it can be applied to any system with high complexities. The technique is effective for small and large-scale systems. As long as the system’s reliability equation can be derived analytically, the model can be used to solve the reliability allocation problem. Different components/subsystems of the system can be selected for optimization. In other words, reliability can be allocated to some or all of the components/subsystems of the system.

The methodology presented is a great tool for aiding engineers in their decision-making. In addition, the approach is very flexible which makes it a great design tool. The parameters of the proposed cost function can be altered, allowing the engineers to investigate different allocation scenarios. Thereafter, reliability and design engineers can decide and plan on how to achieve the assigned minimum required reliabilities for each of the components.

The proposed optimization/allocation model is part of a general system reliability and maintainability software package, called BlockSim™. BlockSim™ is designed to solve for the analytical reliability equation for the system. Therefore, the proposed model can be easily applied.

ACKNOWLEDGMENTS

The author would like to thank Mr. Pantelis Vassiliou for his contribution and support in the research phase of the project which lead to the formulation of this model, as well as Mr. Marios Savva for his efforts in the implementation and solution of the model. In addition the author would like to thank all the previous authors whose research and findings on this subject have inspired the research and development of the proposed model.

REFERENCES


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